Matrix Algebra

Matrices

A matrix is a rectangular array of elements arranged in rows and columns, e.g.:

$$\mathbf{A} = \begin{bmatrix} 6 & 13 \\ 9 & 21 \\ 12 & 5 \end{bmatrix}$$

A here is 3 row by 2 column matrix.

More generally, a matrix of this size can be written as:

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}, \text{ where } \mathbf{A} = [a_{ij}],\ i = 1,2,3;\ j = 1,2.$$

Even more generally, the (r x c) matrix **A** can be written as:

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1j} & \dots & a_{1c} \\ a_{12} & a_{22} & \dots & a_{2j} & \dots & a_{2c} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ a_{i1} & a_{i2} & \dots & a_{ij} & \dots & a_{ic} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ a_{r1} & a_{r2} & \dots & a_{rj} & \dots & a_{rc} \end{bmatrix}$$

where $\mathbf{A} = [a_{ij}]$, $i = 1,\dots,r$; $j = 1,\dots,c$; where r is the number of rows in the matrix and c is the number of columns.

Special Matrices

1) the square matrix ($r = c$), e.g.:

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

2) column vector (an r x 1 matrix)

$$\mathbf{A} = \begin{bmatrix} a_{11} \\ a_{21} \\ \dots \\ a_{r1} \end{bmatrix}$$

3) row vector (a *1 x c* matrix)

$$\mathbf{B} = \begin{bmatrix} b_{11} & b_{12} & \dots & b_{1c} \end{bmatrix}$$

4) a scalar, or 1 *x* 1 matrix: $\mathbf{A} = \begin{bmatrix} a_{11} \end{bmatrix}$

5) the zero matrix:

$$\mathbf{0} = \begin{bmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 \end{bmatrix}$$

6) an r x r diagonal matrix:

$$\mathbf{T} = \begin{bmatrix} t_1 & 0 & \dots & 0 & 0 \\ 0 & t_2 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & t_{r-1} & 0 \\ 0 & 0 & \dots & 0 & t_r \end{bmatrix}$$

7)) vector of ones: **1** = [1 1 1 ... 1]

Matrix operations

1) transposition, e.g. if

$$\mathbf{A} = \begin{bmatrix} 1 & 6 \\ 5 & 3 \\ 0 & 2 \end{bmatrix} \text{ then } \mathbf{A}^T \text{ or } \mathbf{A}' = \begin{bmatrix} 1 & 5 & 0 \\ 6 & 3 & 2 \end{bmatrix}, \text{ or if}$$

$$\mathbf{C} = \begin{bmatrix} 7 \\ 9 \\ 5 \end{bmatrix} \text{ then } \mathbf{C}' = \begin{bmatrix} 7 & 9 & 5 \end{bmatrix},$$

or more generally, if **A** (an *r x c* matrix) is

$$\mathbf{A} = \begin{bmatrix} a_{11} & \dots & a_{1c} \\ \dots & \dots & \dots \\ a_{r1} & \dots & a_{rc} \end{bmatrix}, \text{ then}$$

$$\mathbf{A}' = \begin{bmatrix} a_{11} & \dots & a_{r1} \\ \dots & \dots & \dots \\ a_{1c} & \dots & a_{rc} \end{bmatrix}.$$

A matrix can be said to be symmetric if: $\mathbf{A} = \mathbf{A}'$.

2) equality, **A** = **B**, if $a_{ij} = b_{ij}$, all i and j.

3) addition and subtraction (addition or subtraction of corresponding elements): if

$$\mathbf{A} = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix} \text{ and } \mathbf{B} = \begin{bmatrix} 1 & 2 \\ 2 & 3 \\ 3 & 4 \end{bmatrix}, \text{ then}$$

$$\mathbf{A} + \mathbf{B} = \begin{bmatrix} 1+1 & 4+2 \\ 2+2 & 5+3 \\ 3+3 & 6+4 \end{bmatrix} = \begin{bmatrix} 2 & 6 \\ 4 & 8 \\ 6 & 10 \end{bmatrix}, \text{ and}$$

$$\mathbf{A} - \mathbf{B} = \begin{bmatrix} 1-1 & 4-2 \\ 2-2 & 5-3 \\ 3-3 & 6-4 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 0 & 2 \\ 0 & 2 \end{bmatrix}.$$

More generally,

$$\mathbf{C} = \mathbf{A} + \mathbf{B}, \text{ where } c_{ij} = a_{ij} + b_{ij}$$

4) scalar multiplication (multiplication of each element by the same scalar value): if

$$\mathbf{A} = \begin{bmatrix} 2 & 7 \\ 9 & 3 \end{bmatrix}, \text{ then } 4\mathbf{A} = 4\begin{bmatrix} 2 & 7 \\ 9 & 3 \end{bmatrix} = \begin{bmatrix} 8 & 28 \\ 36 & 12 \end{bmatrix}.$$

More generally,

$$\mathbf{B} = \lambda\mathbf{A}, \text{ where } b_{ij} = \lambda a_{ij}.$$

5) matrix multiplication

$$\text{if } \mathbf{A} = \begin{bmatrix} 2 & 5 \\ 4 & 1 \end{bmatrix}, \text{ and } \mathbf{B} = \begin{bmatrix} 4 & 6 \\ 5 & 8 \end{bmatrix}, \text{ then}$$

$$\mathbf{C} = \mathbf{AB} = \begin{bmatrix} (2\cdot 4 + 5\cdot 5) & (2\cdot 6 + 5\cdot 8) \\ (4\cdot 4 + 5\cdot 1) & (6\cdot 4 + 8\cdot 1) \end{bmatrix}.$$

note that:

a) postmultiplication is not equivalent to premultiplication, i.e.

$$\mathbf{AB} \neq \mathbf{BA};$$

b) matrices must be "conformal" for multiplication, i.e.

$$\underset{(2\times 3)}{\mathbf{D}}\ \underset{(3\times 4)}{\mathbf{E}} = \underset{(2\times 4)}{\mathbf{F}}$$

the number of columns of the first matrix and the number of rows of the second matrix must be equal, and note that the dimensions of the product matrix are given by the number of rows of the first matrix and number of columns of the second:

In general, if

$\mathbf{A} = [a_{ij}],\ i = 1,...,r;\ j = 1,...,c;$

$\mathbf{B} = [b_{ij}],\ i = 1,...,c;\ j = 1,...,s;$ and

$\mathbf{C} = [c_{ij}],\ i = 1,...,r;\ j = 1,...,s;$ then,

$$\mathbf{C} = \mathbf{AB}, \text{ where } c_{ij} = \sum_{k=1}^{c} a_{ik} b_{kj}$$

5) another special matrix is the identity matrix:

$$\mathbf{I} = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 \end{bmatrix}, \text{ and } \mathbf{AI} = \mathbf{IA} = \mathbf{A}.$$

6) inverse matrix (matrix inversion): For a square matrix **A**, the inverse of **A** is the matrix that when premultiplied or postmultiplied by **A** yields the indenty matrix: $\mathbf{A}^{-1}\mathbf{A} = \mathbf{A}\mathbf{A}^{-1} = \mathbf{I}$. Matrix inversion is analogous is some ways to scalar division:

If $(1/x)$ is the inverse of x, then $x \cdot \frac{1}{x} = xx^{-1} = x^{-1}x = 1$.

7) Linear combinations: if

$$\begin{aligned} z_1 &= a_1X_{11} + a_2X_{12} + \dots + a_pX_{1p} \\ z_2 &= a_1X_{21} + a_2X_{22} + \dots + a_pX_{2p} \\ &\dots \\ z_i &= a_1X_{i1} + a_2X_{i2} + \dots + a_pX_{ip} \\ &\dots \\ z_N &= a_1X_{N1} + a_2X_{N2} + \dots + a_pX_{Np} \end{aligned}$$

then this system of equations can be written in more compact form as $\underset{(N\times 1)}{\mathbf{z}} = \underset{(N\times p)(p\times 1)}{\mathbf{X}\quad\mathbf{a}}$, where

$$\underset{(N\times 1)}{\mathbf{z}} = \begin{bmatrix} z_1 \\ z_2 \\ \dots \\ z_N \end{bmatrix}, \quad \underset{(p\times 1)}{\mathbf{a}} = \begin{bmatrix} a_1 \\ a_2 \\ \dots \\ a_p \end{bmatrix}, \text{ and } \underset{(N\times p)}{\mathbf{X}} = \begin{bmatrix} x_{11} & x_{12} & \dots & x_{1p} \\ x_{21} & x_{22} & \dots & x_{2p} \\ \dots & \dots & \dots & \dots \\ x_{N1} & x_{N2} & \dots & x_{Np} \end{bmatrix}.$$

8) eigenvalues and eigenvectors: for a (*p* x *p*) square matrix

R

there is another (*p* x *p*) square matrix

E, such that **RE** = **EV**, where **V** is a (*p* x *p*) diagonal matrix. The first column of these matrices can be written as: $\mathbf{R}\mathbf{e}_1 = \lambda_1 \mathbf{e}_1$, or

$(\mathbf{R} - \lambda\mathbf{I})\mathbf{e}_1 = \mathbf{0}$.

9) quadratic forms: if **A** is an (*n* x *n*) matrix and *x* is an (*n* x 1) column vector, then $Q = \mathbf{x}'\mathbf{A}\mathbf{x} = \mathbf{x}\mathbf{A}\mathbf{x}' = \sum_{i=1}^{n}\sum_{j=1}^{n} x_i a_{ij} x_j$.