Multiple Regression Analysis in Matrix Form

Let

$$\underset{(N\times 1)}{\mathbf{y}} = \begin{bmatrix} y_1 \\ y_2 \\ \dots \\ y_N \end{bmatrix}, \quad \underset{(N\times p+1)}{\mathbf{X}} = \begin{bmatrix} x_{10} & x_{11} & x_{12} & \dots & x_{1p} \\ x_{20} & x_{21} & x_{22} & \dots & x_{2p} \\ \dots & \dots & \dots & \dots & \dots \\ x_{N0} & x_{N1} & x_{N2} & \dots & x_{Np} \end{bmatrix},$$

$$\underset{((p+1)\times 1)}{\mathbf{b}} = \begin{bmatrix} b_0 \\ b_1 \\ \dots \\ b_p \end{bmatrix}, \text{ and } \underset{(N\times 1)}{\mathbf{e}} = \begin{bmatrix} e_1 \\ e_2 \\ \dots \\ e_N \end{bmatrix},$$

and then the standard multiple regression equation:

$$y_i = b_0 x_{i0} + b_1 x_{i1} + b_2 x_{i2} + \dots + b_p x_{ip} + e_i$$

can be written as:

$$\underset{(N\times 1)}{\mathbf{y}} = \underset{(N\times(p\times 1))((p+1)\times 1)}{\mathbf{X}\quad\mathbf{b}} + \underset{(N\times 1)}{\mathbf{e}}$$

The optimization problem in regression analysis is

$$\text{Min S} = \sum_{i=1}^{N} e^2{}_i = \mathbf{e}'\mathbf{e} = (\mathbf{y} - \mathbf{X}\mathbf{b})'(\mathbf{y} - \mathbf{X}\mathbf{b})$$

$$= \mathbf{y}'\mathbf{y} - \mathbf{b}'\mathbf{X}'\mathbf{y} - \mathbf{y}'\mathbf{X}\mathbf{b} + \mathbf{b}'\mathbf{X}'\mathbf{X}\mathbf{b}$$

$$= \mathbf{y}'\mathbf{y} - 2\mathbf{b}'\mathbf{X}'\mathbf{y} + \mathbf{b}'\mathbf{X}'\mathbf{b}.$$

S is mimimized by setting the following partial deriviative to 0:

$$\frac{\partial S}{\partial \mathbf{b}} = -2\mathbf{X}'\mathbf{y} + 2\mathbf{X}'\mathbf{X}\mathbf{b} = 0 \ .$$

Rearranging,

$\mathbf{X}'\mathbf{X}\mathbf{b} = \mathbf{X}'\mathbf{y}$, and so

$$\mathbf{b} = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{y}$$